The Principle of Least Action

In their never-ending search for general principles, from which various laws of Physics could be derived, physicists, and most notably theoretical physicists, have often made use of variational techniques. These usually involved the minimisation of certain quantities. A typical example is *Fermat's principle of least time* where light, propagating between two distant points in space, is postulated to follow the path for which the propagation time is a minimum. A similar principle has been postulated by Hamilton for a material particle following the laws of Classical Physics. The present lecture is devoted to the description and use of such a principle, known as *Hamilton's principle* or *the principle of least action*.
We will start with an analysis of Fermat's principle for light. Once the philosophy behind this idea has been well grasped, we will step into Classical Mechanics and attempt to use the same argument to describe the propagation of a material particle between two distant points in space. In this way the concept of *action* and of *lagrangian* will be automatically introduced. The evolution in time of a moving particle will follow from a variational principle and be described by Euler-Lagrange equations. On more general mechanical systems the same technique can be used if *generalised coordinates* are appropriately introduced; these coordinates take into account the presence of constraints and embed them quite naturally in the formalism developed. Few examples will conclude the lecture.

1 Fermat's principle of least time

This is a simple principle, postulated to govern light propagation. It reads out:

a light ray propagates between two points so as to minimise its travel time

We can formulate this statement quantitatively by using the Calculus of Variations. If the two distant points in space are P_1 and P_2, the principle amounts to finding an extremum for the following integral:

$$I \equiv \int_{P_1}^{P_2} \mathrm{d}t = \int_{P_1}^{P_2} \frac{\mathrm{d}s}{c} \tag{1}$$

where $\mathrm{d}s$ is the line element along the propagation curve and c, the speed of light, depending on position variables.
Let us use equation (1) to show that, in a homogeneous and isotropic material, light propagating between two points P_1 and P_2 follows a straight line. The speed of light, c, is a constant because the material is homogeneous and isotropic. Line element $\mathrm{d}s$ is given by $[(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2]^{1/2}$. In order to carry out the integration we will need to parametrize the curve. This is done in the following way:

$$\left\{ \begin{array}{rcl} x & = & \alpha \\ y & = & f(\alpha) \\ z & = & g(\alpha) \end{array} \right.$$

Given that $\mathrm{d}x/\mathrm{d}\alpha = 1$, $\mathrm{d}y/\mathrm{d}\alpha = f^{'}(\alpha)$ and $\mathrm{d}z/\mathrm{d}\alpha = g^{'}(\alpha)$, integral (1) becomes, in this case:

$$I \equiv \int_{\alpha_1}^{\alpha_2} F[\alpha, f(\alpha), g(\alpha), f^{'}(\alpha), g^{'}(\alpha)]\ \mathrm{d}\alpha$$

$$\Updownarrow$$

$$I \equiv \frac{1}{c}\int_{\alpha_1}^{\alpha_2} \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}y)^2 + (\mathrm{d}z)^2} = \frac{1}{c}\int_{\alpha_1}^{\alpha_2} \sqrt{1 + f'^2(\alpha) + g'^2(\alpha)}\ \mathrm{d}\alpha$$

The above expression includes two functions depending on variable α; the integral extremum will then be found using two Euler-Lagrange equations:

$$\begin{cases} \frac{\partial F}{\partial f} - \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\partial F}{\partial f'} & = & 0 \\ \frac{\partial F}{\partial g} - \frac{\mathrm{d}}{\mathrm{d}\alpha}\frac{\partial F}{\partial g'} & = & 0 \end{cases}$$

$$\Updownarrow$$

$$\begin{cases} f^{'}/\sqrt{1 + f'^2 + g'^2} & = & c_f \\ g^{'}/\sqrt{1 + f'^2 + g'^2} & = & c_g \end{cases}$$

This system can be integrated pretty easily. From it we get:

$$f(\alpha) = K_f\alpha + H_f \quad , \quad g(\alpha) = K_g\alpha + H_g \tag{2}$$

with K_f, K_g, H_f and H_g four new integration constants that can be found using the curve's end points. Equations (2) are parametric equations for a straight line. Thus, Fermat's principle of least time tells us that in a homogeneous, isotropic medium light always propagates along a straight line.

Using Fermat's principle it is also possible to derive Snell's law of refraction in an easy way, as explained in several textbooks. Here we will attempt the solution of the more general problem of finding light trajectory in a medium for which the refractive index is given by an analytical expression. The *refractive index* of a given medium is defined as the ratio between light velocities in vacuum and that medium:

$$n \equiv \frac{c}{v} \tag{3}$$

We know that c is a constant. Using (3) Fermat's principle can be seen as a variational problem on the following integral:

$$I \equiv \frac{1}{c}\int_{P_1}^{P_2} n\ \mathrm{d}s \tag{4}$$

Once n is given as a function of space coordinates, Euler-Lagrange equations can be found, and the desired light trajectory found. Consider for example a refractive index depending on the z-coordinate only, according to the following formula:

$$n = n(z) = n_0 - \lambda z \tag{5}$$

Given that n depends exclusively on z, light behaviour will be the same on any plane containing the z-axis. Let us, therefore, focus on one of these planes, plane (x, z). On it the line element will be $\mathrm{d}s = \sqrt{(\mathrm{d}x)^2 + (\mathrm{d}z)^2}$. Integral (4) can in this case be re-written as:

$$I = \frac{1}{c}\int_{P_1}^{P_2} n(z)\sqrt{1 + x'^2}\ \mathrm{d}z$$

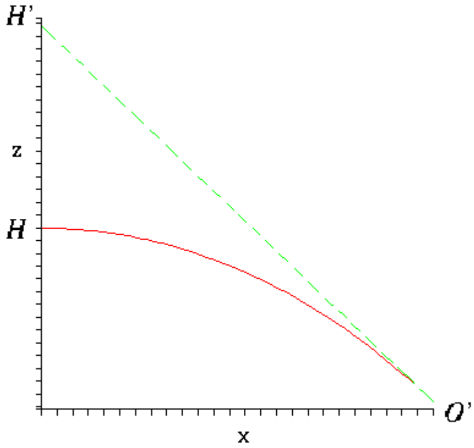

Figure 1: Light sent by a building in H appears as sent by $H^{'}$, due to the variation of the refractive index with the altitude z.

where it is obvious that we have chosen to express x as a function of z. To minimise integral I we have to solve the following Euler-Lagrange equation:

$$\frac{n(z)x^{'}}{\sqrt{1+x^{'2}}} = k \qquad \Rightarrow \qquad x^{'} = \frac{k}{\sqrt{n^2(z)-k^2}}$$

where k is essentially an integration constant. We can now replace expression (5) for $n(z)$ and, after the integration, obtain:

$$z = \frac{1}{\lambda}\left[n_0 - k\cosh\left(\lambda\frac{h-x}{k}\right)\right] \tag{6}$$

where h is another integration constant. h and k can be determined once initial and final propagations points are given. A typical situation is when light from a tall building H is received by an observer at $O^{'}$ (see Figure 1). Due to the bending caused by the variable refractive index, the observer will see light as if it were send by the taller building in $H^{'}$. The same type of variation in the refractive index is responsible for mirages.

2 Hamilton's principle

The propagation of light in space can be described using the least time principle. Light rays are found as solutions of Euler-Lagrange equations. Is it feasible to extend a similar approach to Classical Mechanics? Can we find the equations governing, for instance, a particle motion simply as Euler-Lagrange equations? The answer to these questions, in many physical cases, is 'yes'. Let us consider the simplest of all mechanical systems, the free particle in one dimension. Its motion is described by the following equation:

$$m\ddot{x} = 0 \tag{7}$$

where the double dot on x denotes a second derivative with respect to time. The solution to equation (7) is $x(t) = vt + x_0$, i.e. the particle proceeds along a straight line passing through x_0 at $t = 0$, with velocity v. It is simple to verify that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial\dot{x}}\left(\frac{1}{2}m\dot{x}^2\right)\right] = m\ddot{x}$$

The above expression has the same form of the left-hand side of a Euler-Lagrange equation like:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial F(t,\dot{x})}{\partial \dot{x}}\right] = 0$$

Thus, equation (7) can be derived by requiring that functional:

$$S \equiv \int_{\gamma} \frac{1}{2} m\dot{x}^2 \ \mathrm{d}t \tag{8}$$

be a minimum (γ is the integration curve between initial and final points in the plane (t,x)). Let us go a step forward and consider the particle immersed in a conservative field. The equation of motion will be, in this case:

$$m\ddot{x} = F = -\frac{\partial V}{\partial x} \qquad \Rightarrow \qquad -\frac{\partial V}{\partial x} - m\ddot{x} = 0 \tag{9}$$

where we have used the fact that $F = -\partial V/\partial x$ for a conservative force. Equation (9) is slightly more complicated than equation (7). It can be re-written as:

$$-\frac{\partial V}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\left[\frac{\partial}{\partial \dot{x}}\left(\frac{1}{2}m\dot{x}^2\right)\right] = 0$$

This equation can be formally made simpler by defining the following function:

$$L \equiv T - V \equiv \frac{1}{2}m\dot{x}^2 - V \tag{10}$$

With this new definition, equation (9) becomes:

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = 0$$

and it is equivalent to the Euler-Lagrange equation deriving from the following functional:

$$S \equiv \int_{\gamma} L(t,x,\dot{x}) \ \mathrm{d}t \tag{11}$$

Integrand L is called *lagrangian* of the system. Dimensionally it is an energy, because it is defined as the difference between kinetic and potential energies. Functional (11) has the dimensions of energy×time, and is known as the *action* of the system. In the very simple case just treated we have shown that the equation of motion for a particle in one dimension can be derived from the requirement that the path followed by the particle makes the action a minimum. This is a first, crude formulation of *Hamilton's principle*. It is still a valid formulation when three dimensions are considered. The lagrangian is the difference between kinetic and potential energies:

$$L(t,x,y,z,\dot{x},\dot{y},\dot{z}) = \frac{1}{2}m(\dot{x}^2+\dot{y}^2+\dot{z}^2) - V(t,x,y,z)$$

With three variables and three derivatives, three Euler-Lagrange equations will be formed:

$$\frac{\partial L}{\partial x} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = 0 \ , \ \frac{\partial L}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{y}} = 0 \ , \ \frac{\partial L}{\partial z} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{z}} = 0$$

which coincides with the equations of motion:

$$m\ddot{x} = F_x \ , \ m\ddot{y} = F_y \ , \ m\ddot{z} = F_z$$

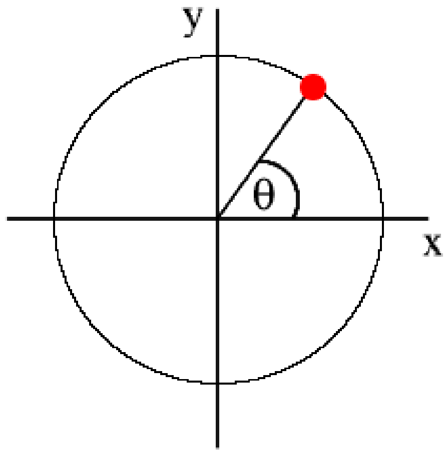

Figure 2: A bead sliding through a wire hoop. *theta* is the generalised coordinate which defines completely the system.

When more than a particle is considered, things are no more difficult to handle than the case just considered. Simply more variables will be included in the expression for the lagrangian, and several Euler-Lagrange equations will follow from the action integral.
In Classical Mechanics things can become very complicated when constraints are introduced. Consider, for instance, a bead sliding without friction in a wire hoop of radius R, lying flat on the plane $z = 0$ (see Figure 2). The problem can be studied within a newtonian framework, by setting up equations like (9) for both variables x and y. There will be x- and y- components for external forces. Other forces, arising from the bead being compelled to follow a circular trajectory, will have to be considered in addition to the external ones. We realise pretty soon, though, that the problem is inherently simpler than what appears from the formalism just described. First of all, one variable, the angle θ, and not two is sufficient to describe the bead's motion. Second, we should not, really, need to consider additional forces, as they are only a spurious consequence of the circular constraint. Thus, a variable θ can be used to describe our system. This will, automatically, take the fictitious forces into account, as θ is a circular variable. But, what will, at this point, be the equation of motion? It can be shown that these can again be derived by Hamilton's principle. A lagrangian $L(t, \theta, \dot{\theta})$ will be set up, first, and an Euler-Lagrange equation will follow. θ cannot be considered a coordinate like x or y, but it is nonetheless a valid variable to describe the bead's dynamic evolution. To variables like θ, that are not conventional coordinates, but fully describe the dynamic evolution of a mechanical system, the name *generalised coordinates* has been assigned. In general, a mechanical system with n degrees of freedom can be entirely defined using n generalised coordinates. It is customary to use symbols q_1, q_2,...,q_n to indicate these variables, while $\dot{q}_1$, $\dot{q}_2$,...,$\dot{q}_n$, are their time derivatives. The lagrangian is the difference between kinetic and potential energies expressed using generalised coordinates,

$$L \equiv T - V = L(t, q_1, q_2, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_n)$$

and the action, S, is the following integral:

$$S \equiv \int_\gamma L(t, q_1, q_2, ..., q_n, \dot{q}_1, \dot{q}_2, ..., \dot{q}_n) \, \mathrm{d}t \tag{12}$$

where γ is a curve in n-dimensional space. From Hamilton's principle, alternatively called the *least action principle*, a set of n Euler-Lagrange equations is derived,

$$\frac{\partial L}{\partial q_j} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_j} = 0 \qquad j = 1, 2, ..., n \tag{13}$$

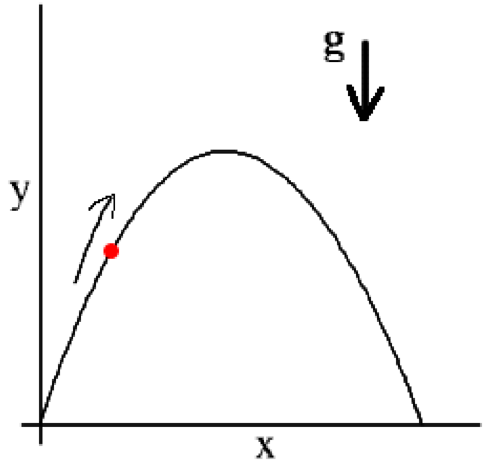

Figure 3: Motion of a particle in a uniform gravitational field.

Let us consider the bead-in-the-hoop example. First we have to express x, y and z as functions of the generalised coordinate θ:

$$\begin{cases} x &= R\cos(\theta) \\ y &= R\sin(\theta) \\ z &= 0 \end{cases}$$

The kinetic energy will , then, be given by:

$$T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) = \frac{1}{2}mR^2\dot{\theta}^2$$

while external forces will be described by a potential energy depending only on time and positions:

$$V = V(t, \theta)$$

The lagrangian for this system will, accordingly, be:

$$L = T - V = \frac{1}{2}mR^2\dot{\theta}^2 - V(t, \theta)$$

This lagrangian depends on just one generalised coordinate, θ; the system's dynamic evolution will, then, be described by just one Euler-Lagrange equation:

$$\frac{\partial L}{\partial \theta} - \frac{\mathrm{d}}{\mathrm{d}\theta}\frac{\partial L}{\partial \dot{\theta}} = 0 \qquad \Rightarrow \qquad mR^2\ddot{\theta} = -\frac{\partial V}{\partial \theta}$$

3 Applications of Hamilton's principle

EXAMPLE 1. PROJECTILE MOTION
Consider a first easy application of Hamilton's principle to the motion of a body under the influence of gravity on Earth's surface. This can be approximated by the motion of a particle in a uniform gravitational field (see Figure 3). The system evolves in a plane, and there are no constraints. We will, then, need two generalised coordinates; in this case they coincide with the cartesian coordinates x and y. Kinetic and potential energies are given by:

$$T = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) \quad , \quad V = mgy$$

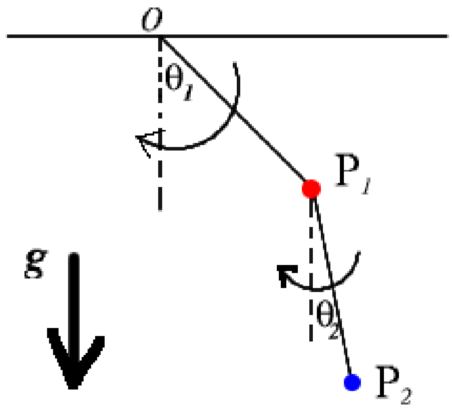

Figure 4: Motion of a particle in a uniform gravitational field.

where m is the particle mass and g gravity acceleration. The lagrangian is therefore defined as:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) - mgy$$

Two degrees of freedom imply two Euler-Lagrange equations. These are readily written as:

$$\begin{cases} m\ddot{x} &= 0 \\ m\ddot{y} &= -mg \end{cases} \quad \Rightarrow \quad \begin{cases} \ddot{x} &= 0 \\ \ddot{y} &= -g \end{cases}$$

The above equations are clearly recognised as describing a projectile moving in a uniform gravitational field.

<u>EXAMPLE 2</u>. DOUBLE PENDULUM

We will now examine a problem where the presence of constraints makes Newton's equation difficult to apply. The introduction of generalised coordinates which take all constraints into account, and the subsequent formulation using Hamilton's principle, enable us to lay down dynamics equations quite straightforwardly. In Figure 4 the double pendulum is described as consisting of a particle $\mathcal{P}_1$, of mass m_1, connected to the origin of the coordinate system through a cord of length ℓ_1; another particle, $\mathcal{P}_2$ of mass m_2, is connected to $\mathcal{P}_1$ through a cord of length ℓ_2. The only external force is gravity, which acts on both particles. Due to the various constraints, though, tension forces have to be taken into account to properly define Newton's law on both particles. Furthermore, the two particles need apparently four coordinates to be described, i.e. the system seems to possess four degrees of freedom. We immediately realise, though, that only the two angles θ_1 and θ_2 are enough to describe the evolution, here. And, by choosing these two angles as generalised coordinates, we automatically take all constraint forces into account. Let us, therefore, express the four coordinates x_1, y_1, x_2 and y_2 as functions of θ_1 and θ_2:

$$\begin{cases} x_1 &= \ell_1 \sin(\theta_1) \\ y_1 &= -\ell_1 \cos(\theta_1) \end{cases}, \quad \begin{cases} x_2 &= x_1 + \ell_2 \sin(\theta_2) &= \ell_1 \sin(\theta_1) + \ell_2 \sin(\theta_2) \\ y_2 &= y_1 - \ell_2 \cos(\theta_2) &= -[\ell_2 \cos(\theta_1) + \ell_2 \cos(\theta_2)] \end{cases} \tag{14}$$

Now we are able to compute the kinetic energy as a function of generalised coordinates. Deriving relations (14) with respect to time yields:

$$\begin{cases} \dot{x}_1 &= \ell_1 \dot{\theta}_1 \cos(\theta_1) \\ \dot{y}_1 &= \ell_1 \dot{\theta}_1 \sin(\theta_1) \end{cases}, \quad \begin{cases} \dot{x}_2 &= \ell_1 \dot{\theta}_1 \cos(\theta_1) + \ell_2 \dot{\theta}_2 \cos(\theta_2) \\ \dot{y}_2 &= \ell_1 \dot{\theta}_1 \sin(\theta_1) + \ell_2 \dot{\theta}_2 \sin(\theta_2) \end{cases}$$

Thus we have for the kinetic energy, using the above expressions:

$$T = \frac{1}{2}\left[m_1(\dot{x_1}^2 + \dot{y_1}^2) + m_2(\dot{x_2}^2 + \dot{y_2}^2)\right]$$

$$\Downarrow$$

$$T = \frac{1}{2}\left\{m_1\ell_1^2\dot{\theta_1}^2 + m_2\left[\ell_1^2\dot{\theta_1}^2 + \ell_2^2\dot{\theta_2}^2 + 2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right]\right\} \tag{15}$$

For the potential energy we only need to use relations (14):

$$V = m_1 g y_1 + m_2 g y_2 = g(m_1 y_1 + m_2 y_2)$$

$$\Downarrow$$

$$V = g\left\{-m_1\ell_1\cos(\theta_1) - m_2\left[\ell_1\cos(\theta_1) + \ell_2\cos(\theta_2)\right]\right\} \tag{16}$$

The lagrangian of this system is, as usual, the difference between kinetic and potential energies. Using expressions (15) and (16) we obtain:

$$\begin{aligned} L \; = \; & \frac{1}{2}\left\{m_1\ell_1^2\dot{\theta_1}^2 + m_2\left[\ell_1^2\dot{\theta_1}^2 + \ell_2^2\dot{\theta_2}^2 + 2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)\right]\right\} + \\ & g\left\{-m_1\ell_1\cos(\theta_1) - m_2\left[\ell_1\cos(\theta_1) + \ell_2\cos(\theta_2)\right]\right\} \end{aligned} \tag{17}$$

From this lagrangian we can easily compute partial derivatives with respect to θ_1, θ_2, $\dot{\theta}_1$, and $\dot{\theta}_2$. These will be successively fed into two Euler-Lagrange equations, which are the dynamical equation we were looking for. We will not attempt to solve this nonlinear system of differential equations here (it can be integrated when small oscillations are considered). The important point we wanted to focus on was that Hamilton's principle, and the related variational calculus apparatus, constitute a powerful formalism to handle mechanical systems with constraints.

EXAMPLE 3. CHARGED PARTICLE IN AN ELECTROMAGNETIC FIELD
Hamilton's principle can be applied also to physical systems which include laws from branches of Physics different from Classical Mechanics, like for instance Electromagnetism. It can, indeed, be applied to all those cases where an action integral is defined. This, in turn, implies the possibility of specifying a lagrangian as function of generalised coordinates. Considering a particle immersed in some external field of forces, the crucial step will be to find the right potential energy. For the case of a particle of charge e, immersed in an electromagnetic field, the potential energy is given by the following expression:

$$V = e\left[\Phi(t,x,y,z) - \frac{1}{c}\dot{\mathbf{r}}\cdot\mathbf{A}(t,x,y,z)\right] \tag{18}$$

where Φ and $\mathbf{A}$ are the scalar and vector potentials respectively, while $\mathbf{r} \equiv (x,y,z)$. The lagrangian function is, therefore, defined in the following way:

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - e\left[\Phi - \frac{1}{c}\left(\dot{x}A_x + \dot{y}A_y + \dot{z}A_z\right)\right] \tag{19}$$

where Φ, A_x, A_y, A_z depend on t, x, y, and z. Let us write down the Euler-Lagrange equation for the first coordinate, x. First we have to determine two partial derivatives:

$$\frac{\partial L}{\partial x} = -e\frac{\partial\Phi}{\partial x} + \frac{e}{c}\left(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x}\right)$$

$$\frac{\partial L}{\partial \dot{x}} = m\dot{x} + \frac{e}{c}A_x$$

Second, we need to compute the derivative of $\partial L/\partial \dot{x}$ with respect to time:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{x}} = m\ddot{x} + \frac{e}{c}\left(\frac{\partial A_x}{\partial t} + \dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_x}{\partial y} + \dot{z}\frac{\partial A_x}{\partial z}\right)$$

At this point the Euler-Lagrange equation is readily written as:

$$-e\frac{\partial \Phi}{\partial x} + \frac{e}{c}\left(\dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_y}{\partial x} + \dot{z}\frac{\partial A_z}{\partial x}\right) - m\ddot{x} - \frac{e}{c}\left(\frac{\partial A_x}{\partial t} + \dot{x}\frac{\partial A_x}{\partial x} + \dot{y}\frac{\partial A_x}{\partial y} + \dot{z}\frac{\partial A_x}{\partial z}\right) = 0$$

$$\Downarrow$$

$$m\ddot{x} = e\left(-\frac{\partial \Phi}{\partial x} - \frac{1}{c}\frac{\partial A_x}{\partial t}\right) + \frac{e}{c}\left[\dot{y}\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right) + \dot{z}\left(\frac{\partial A_z}{\partial x} - \frac{\partial A_x}{\partial z}\right)\right]$$

The expression just obtained is the x-component of the following vectorial equation:

$$m\ddot{\mathbf{r}} = e\left(-\nabla\Phi - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t}\right) + \frac{e}{c}\dot{\mathbf{r}} \times (\nabla \times \mathbf{A}) \tag{20}$$

with $\times$ denoting cross product. It is left as an exercise to verify that the Euler-Lagrange equations for the y- and z-component coincide with the y- and z-component of equation (20). Now, we know that the electric and magnetic fields are defined using scalar and vector potentials as:

$$\begin{cases} \mathbf{E} &= -\nabla\Phi - (1/c)\partial\mathbf{A}/\partial t \\ \mathbf{B} &= \nabla \times \mathbf{A} \end{cases}$$

Thus, equation (20) can be re-written as:

$$m\ddot{\mathbf{r}} = e\mathbf{E} + \frac{e}{c}\dot{\mathbf{r}} \times \mathbf{B} \tag{21}$$

This last equation is easily recognizable as the one describing the motion of a charged particle in an electromagnetic field, where both electrostatic and Lorentz forces are experienced. This is a further proof that expression (19) is the correct expression for the lagrangian.