Laplace's Equation in Spherical Coordinates and Legendre's Equation (I)

Legendre's equation arises when one tries to solve Laplace's equation in spherical coordinates, much the same way in which Bessel's equation arises when Laplace's equation is solved using cylindrical coordinates. In this lecture we will introduce Legendre's equation and provide solutions physically meaningful in form of converging series. We will delay the full treatment of Laplace's equation in spherical coordinates to the end of the lecture, once the tools needed to solve it have been thoroughly introduced.

1 Power series solution of Legendre's equation

Legendre's equation is one of the important equations in mathematical physics. It is usually written in the following form

$$(1-x^2)f''(x) - 2xf'(x) + \alpha f(x) = 0 \tag{1}$$

where α is a real constant. Let us set ourselves to solve equation (1) using a power series expansion in the neighborhood of $x = 0$, which is a regular point for the equation. We start by postulating the following form for $f(x)$,

$$f(x) = \sum_{n=0}^{+\infty} a_n x^n \tag{2}$$

Next, we need first and second derivative of expression (2). These are readily obtained as

$$f'(x) = \sum_{n=1}^{+\infty} n a_n x^{n-1} \ , \ \ f''(x) = \sum_{n=2}^{+\infty} n(n-1) a_n x^{n-2} \tag{3}$$

(observe that the above summations runs from 1 and 2, rather than from 0). All is left to do, now, is to replace quantities (2) and (3) into equation (1). The resulting expression yields

$$\sum_{n=2}^{+\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{+\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{+\infty} n a_n x^n + \alpha \sum_{n=0}^{+\infty} a_n x^n = 0$$

or, by replacing n with $n+2$ in the first summation,

$$\sum_{n=0}^{+\infty}(n+2)(n+1)a_{n+2}x^n - \sum_{n=2}^{+\infty} n(n-1)a_n x^n - 2\sum_{n=1}^{+\infty} n a_n x^n + \alpha \sum_{n=0}^{+\infty} a_n x^n = 0$$

In the second summation we can start counting from 0, because terms with $n = 0$ and $n = 1$ are null. Similarly, we start counting from 0 in the third summation. This way all summations run from 0 to $+\infty$. Therefore we can use a single summation symbol for the whole equation:

$$\sum_{n=0}^{+\infty} \{(n+2)(n+1)a_{n+2} - [n(n-1) + 2n - \alpha]\, a_n\}\, x^n = 0$$

Equating the coefficient of each power of x to zero (and rearranging the resulting expression), we obtain, finally,

$$a_{n+2} = \frac{n(n+1) - \alpha}{(n+1)(n+2)} a_n \quad , \quad n = 0, 1, 2, 3, \ldots \tag{4}$$

From (4) one can calculate all coefficients a_n, once a_0 and a_1 are known. More specifically, it is quite evident that a_0 will generate all coefficients with even index $(a_2, a_4, a_6, \ldots)$, while a_1 will generate all coefficients with odd index $(a_3, a_5, a_7, \ldots)$. For even coefficients we have:

$$\begin{aligned}
a_2 &= -\frac{\alpha}{1 \cdot 2} a_0 \\
a_4 &= -\frac{\alpha - 2 \cdot 3}{3 \cdot 4} a_2 = (-1)^2 \frac{(\alpha - 2 \cdot 3)\alpha}{4!} a_0 \\
a_6 &= -\frac{\alpha - 4 \cdot 5}{5 \cdot 6} a_4 = (-1)^3 \frac{(\alpha - 4 \cdot 5)(\alpha - 2 \cdot 3)\alpha}{6!} a_0 \\
&\cdots \\
&\cdots \\
a_{2r} &= (-1)^r \frac{[\alpha - (2r-1)(2r-2)][\alpha - (2r-3)(2r-4)] \cdots \alpha}{(2r)!} a_0
\end{aligned} \tag{5}$$

with $r = 0, 1, 2, \ldots$.For odd coefficients, similarly,

$$\begin{aligned}
a_3 &= -\frac{\alpha - 1 \cdot 2}{2 \cdot 3} a_1 \\
a_5 &= -\frac{\alpha - 3 \cdot 4}{4 \cdot 5} a_3 = (-1)^2 \frac{(\alpha - 3 \cdot 4)(\alpha - 1 \cdot 2)}{5!} a_1 \\
a_7 &= -\frac{\alpha - 5 \cdot 6}{6 \cdot 7} a_5 = (-1)^3 \frac{(\alpha - 5 \cdot 6)(\alpha - 3 \cdot 4)(\alpha - 1 \cdot 2)}{7!} a_1 \\
&\cdots \\
&\cdots \\
a_{2r+1} &= (-1)^r \frac{[\alpha - 2r(2r-1)][\alpha - (2r-2)(2r-3)] \cdots (\alpha - 1 \cdot 2)}{(2r+1)!} a_1
\end{aligned} \tag{6}$$

again, with $r = 0, 1, 2, \ldots$. Given that the coefficients with even index are only dependent on a_0, and those with odd index are only dependent on a_1, coefficients (5) and (6) represent, actually, two independent solutions for Legendre's equation (one only contains even power of x, the other only odd powers). Let us find the radius of convergence for these two series solutions. A series converges if the ratio of two consecutive terms converges to a number smaller than 1 for n getting bigger and bigger. For the solution with even indices we have to compute, then, the following limit:

$$R = \lim_{r \to +\infty} \frac{|a_{2r+2}| x^{2r+2}}{|a_{2r}| x^{2r}} = \lim_{r \to +\infty} \frac{|\alpha - 2r(2r+1)|}{|(2r+2)(2r+1)|} x^2 = x^2$$

Now, $x^2 < 1$ when $-1 < x < 1$. Therefore the solution described by the series with even coefficients converges for all values of x between -1 and +1. The same can be verified for the series with odd coefficients. Given that in the majority of the applications of interest to a physicist

the variable x is, in fact, a cosine, the convergence interval just found is all we need to explore useful solutions.
To summarize, if the following functions are defined,

$$f_E(\alpha; x) \equiv 1 + \sum_{r=1}^{+\infty} (-1)^r \frac{[\alpha - (2r-1)(2r-2)][\alpha - (2r-3)(2r-4)] \cdots \alpha}{(2r)!} x^{2r} \quad (7)$$

$$f_O(\alpha; x) \equiv x \left\{ 1 + \sum_{r=1}^{+\infty} (-1)^r \frac{[\alpha - 2r(2r-1)][\alpha - (2r-2)(2r-3)] \cdots (\alpha - 2 \cdot 1)}{(2r+1)!} x^{2r} \right\} \quad (8)$$

then the general solution of Legendre's equation, converging in the open interval $(-1, 1)$, is:

$$f(x) = a_0 f_E(\alpha; x) + a_1 f_O(\alpha; x) \quad (9)$$

with a_0 and a_1 two constants which will assume specific values for particular solutions.

<u>EXAMPLE 1</u>.
Find the solution of equation,

$$(1 - x^2) y^{''} - 2xy' - 3/2y = 0$$

subject to the following initial conditions,

$$y(0) = 1 \qquad , \qquad y^{'}(0) = 0$$

<u>Solution</u>.
The equation is a Legendre's equation with $\alpha = -3/2$. Therefore its general solution is,

$$y(x) = a_0 f_E(-3/2; x) + a_1 f_O(-3/2; x)$$

Now, $f_E(-3/2; 0) = 1$, while $f_O(-3/2; 0) = 0$. And $f_E^{'}(-3/2; 0) = 0$, while $f_O^{'}(-3/2; 0) = 1$. We will have, then, $y(0) = a_0$ and $y^{'}(0) = a_1$. The initial conditions, thus, yield $a_0 = 1$ and $a_1 = 0$, and the soulution sought for is:

$$y(x) = f_E(-3/2; x) \equiv 1 + \sum_{r=1}^{+\infty} (-1)^r \frac{[-3/2 - (2r-1)(2r-2)] \cdots (-3/2)}{(2r)!} x^{2r}$$

certainly converging for $-1 < x < 1$.

2 Legendre polynomials

Solution (9) converges for all values inside the interval between -1 and +1, but diverges or it is not defined at $x = \pm 1$. This feature makes it a weak candidate for physical problems where, most of the time, finite solutions are needed. For instance, we know that in several applications x is a cosine, and a cosine is defined in the full interval $[-1, 1]$, including ± 1; they correspond to values 0 and π of the angle, values for which the physical applications have, quite possibly, regular and finite solutions. Therefore, we need to find solutions of Legendre's equation which are defined and converge in the full interval $[-1, 1]$.
By looking at recurrence relation (4) we notice that when α equals a positive integer (or zero), which can be expressed as $l(l+1)$ $(l \geq 0)$, then all coefficients a_n with $n > l$ are zero. For example, the following Legendre's equation,

$$(1 - x^2) f^{''}(x) - 2x f^{'}(x) + 20 f(x) = 0$$

has $\alpha = 4(4+1)$. The recurrence formula for even indexes reads, in this case,

$$a_{n+2} = \frac{n(n+1) - 4\cdot 5}{(n+1)(n+2)} a_n$$

i.e.,

$$\begin{aligned}
a_2 &= \frac{0\cdot 1 - 4\cdot 5}{1\cdot 2} a_0 = -10 a_0 \\
a_4 &= \frac{2\cdot 3 - 4\cdot 5}{3\cdot 4} a_2 = \left(-\frac{7}{6}\right)(-10 a_0) = \frac{35}{3} a_0 \\
a_6 &= \frac{4\cdot 5 - 4\cdot 5}{5\cdot 6} a_4 = 0\cdot \frac{35}{3} a_0 = 0 \\
a_8 &= \frac{6\cdot 7 - 4\cdot 5}{7\cdot 8} a_6 = 0 \\
a_{10} &= \frac{8\cdot 9 - 4\cdot 5}{9\cdot 10} a_8 = 0 \\
a_{12} &= 0 \\
a_{14} &= 0 \\
&\dots \\
&\dots
\end{aligned}$$

Thus, we see that, given $l = 4$, $a_n = 0$ for $n > 4$. We could have chosen l equal to an odd integer, and have all odd coefficients with index greater than that integer equal to zero. The point here is that, once α is selected as an integer of the form $l(l+1)$, solution (9) can be turned into a finite function for the whole interval $[-1, 1]$ by appropriately setting $a_0 = 0$ if l is odd, or $a_1 = 0$ if l is even. This function is a polynomial whose properties will be studied, next.
First of all, given a specific value for l, recurrence relation (4) assumes, after a few factorizations, the following form:

$$a_{n+2} = -\frac{(l-n)(l+n+1)}{(n+1)(n+2)} a_n \quad , \quad n = 0, 1, 2, \dots, l \tag{10}$$

Through equation (10) we are going to generate only a finite number of terms, as we are building coefficients for a finite polynomial. Let us consider the term with $n = l-2$. From (10):

$$a_l = -\frac{2(2l-1)}{l(l-1)} a_{l-2} \Rightarrow a_{l-2} = -\frac{l(l-1)}{2(2l-1)} a_l$$

Now, it is customary for special functions, like the polynomials we are trying to build here, to fix all constants, so to standardize all results. The general solution will still be expressed as (9), with two different constants. Let us, therefore, adopt the following definitions:

$$a_0 = 1 \ , \ a_n = \frac{(2n)!}{2^n (n!)^2} \quad , \quad n = 1, 2, 3, \dots \tag{11}$$

We will have, thus,

$$a_{l-2} = -\frac{l(l-1)}{2(2l-1)} \frac{(2l)!}{2^l (l!)^2} = -\frac{(2l-2)!}{2^l (l-2)!(l-1)!}$$

Let us now consider, in (10), $n = l-4$:

$$a_{l-2} = -\frac{4(2l-3)}{(l-3)(l-2)} a_{l-4} \Rightarrow a_{l-4} = -\frac{(l-3)(l-2)}{4(2l-3)} a_{l-2}$$

or, by using the previous result,

$$a_{l-4} = (-1)^2 \frac{(l-3)(l-2)}{4(2l-3)} \frac{(2l-2)!}{2^l(l-2)!(l-1)!} = (-1)^2 \frac{(2l-4)!}{2^l 2(l-2)!(l-4)!}$$

Proceeding along similar lines, we obtain, for $n = l - 6$,

$$a_{l-6} = (-1)^3 \frac{(2l-6)!}{2^l 3!(l-3)!(l-6)!}$$

and, in general,

$$a_{l-2m} = (-1)^m \frac{(2l-2m)!}{2^l m!(l-m)!(l-2m)!}$$

Using definition (11), we have built a class of particular solutions of Legendre's equation with $\alpha = l(l+1)$, where l is a positive or zero integer. These are called *Legendre polynomials*, and are defined according to the following formula:

$$P_l(x) = \sum_{m=0}^{M} (-1)^m \frac{(2l-2m)!}{2^l m!(l-m)!(l-2m)!} x^{l-2m} \tag{12}$$

where $M = l/2$ or $M = (l-1)/2$, whichever is an integer.

<u>EXAMPLE 2</u>.
Write the first four Legendre polynomials, using formula (12).
<u>Solution</u>
$l = 0$.

$$P_0(x) = 1$$

$l = 1$.

$$P_1(x) = x$$

$l = 2$.

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$l = 3$.

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

A plot of the first four Legendre polynomials is shown at Figure 1. It is interesting to observe that all polynomials pass through point (1,1). This is, in fact, due to the standardization choice (11), according to which $P_l(1) = 1$, for any value of l.

<u>EXAMPLE 3</u>.
Find the solution to the following Legendre's equation:

$$(1 - x^2)y'' - 2xy' + 12y = 0,$$

with $y(0) = 0$ and $y'(0) = 1$.
<u>Solution</u>
In this case α is an integer of the form $l(l+1)$, with $l = 3$. Therefore, the general solution is:

$$y(x) = a_0 f_E(12; x) + BP_3(x)$$

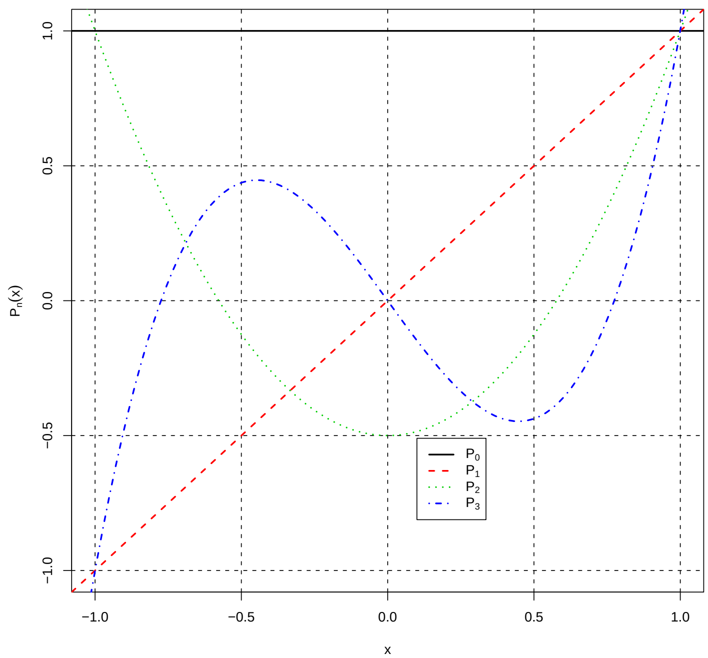

Figure 1: Plot of the first four Legendre polynomials.

where a_0 is the same constant used in equation (9), while B is a new constant which replaces a_1, due to the standardization chosen for Legendre polynomial $P_3(x)$. Given that $f_E(12;0) = 1$, $P_3(0) = 0$, $f_E^{'}(12;0) = 0$, and $P^{'}(0) = -3/2$, the initial conditions yield,

$$a_0 = 0 \quad , \quad -\frac{3}{2}B = 1 \;\Rightarrow\; B = -\frac{2}{3}$$

The solution we were looking for is, thus,

$$y(x) = -\frac{2}{3}P_3(x)$$

This solution is simply a polynomial, finite in the whole interval $[-1, 1]$. Initially the general solution had not this property but, thanks to the initial conditions, the particular solution turned into a finite solution. Very often solutions to a Legendre's equation to be used in a physical problem will be coupled to initial or boundary conditions that will cause them to become finite, tractable solutions.

3 Legendre functions of the second kind

Legendre polynomials can be easily derived from functions $f_E(l(l+1);x)$ and $f_O(l(l+1);x)$, simply multiplying them by an appropriate constant. More specifically, we can obtain $P_n(x)$ for even values of n using the following formula,

$$P_n(x) = (-1)^{n/2}\frac{n!}{2^n[(n/2)!]^2}f_E(n(n+1);x) \quad \text{, for n even} \tag{13}$$

and $P_n(x)$ for odd values of n using this other formula,

$$P_n(x) = (-1)^{(n-1)/2}\frac{n!}{2^{n-1}\{[(n-1)/2]!\}^2}f_O(n(n+1);x) \quad \text{, for n odd} \tag{14}$$

Let us test, for instance, formula (14) for $n = 3$. We have,

$$P_3(x) = -\frac{3!}{4} f_O(12; x) = -\frac{3}{2} x \left\{ 1 + \sum_{r=1}^{+\infty} (-1)^r \frac{[12 - 2r(2r-1)][12 - (2r-2)(2r-3)] \cdots (12-2)}{(2r+1)!} x^{2r} \right\}$$

$$\Downarrow$$

$$P_3(x) = -\frac{3}{2} x \left\{ 1 - \frac{12 - 2 \cdot 1}{3!} x^2 + \frac{(12 - 4 \cdot 3)(12 - 2 \cdot 1)}{5!} x^4 - \frac{(12 - 6 \cdot 5)(12 - 4 \cdot 3)(12 - 2 \cdot 1)}{7!} x^6 + \cdots \right\}$$

$$\Downarrow$$

$$P_3(x) = -\frac{3}{2} x \left\{ 1 - \frac{12 - 2 \cdot 1}{3!} x^2 \right\} = \frac{1}{2}(5x^3 - 3x),$$

which is the correct expression for $P_3(x)$. As it happens, an integer value $12 = 3(3+1)$ determined the series truncation, and the final result was a polynomial. In order to obtain this one should be careful to select f_E for even n, and f_O for odd n. If $f_E(n(n+1); x)$ was written for odd values of n, or $f_O(n(n+1); x)$ was written for even values of n, infinite series, rather than polynomials, would be produced, although they would still represent functions converging in the open interval $(-1, 1)$. These functions are known as *Legendre functions of the second kind*, and are standardised and defined as follows:

$$Q_n(x) = (-1)^{n/2} \frac{[(n/2)!]^2 2^n}{n!} f_O(n(n+1); x) \quad \text{, for n even} \tag{15}$$

$$Q_n(x) = (-1)^{(n+1)/2} \frac{\{[(n-1)/2]!\}^2 2^{n-1}}{n!} f_E(n(n+1); x) \quad \text{, for n odd} \tag{16}$$

EXAMPLE 4.
Using formulas (15) and (16), compute $Q_0(x)$ and $Q_1(x)$.
Solution
Let us start with $n = 0$. Being this an even number, we will have to use formula (15),

$$Q_0(x) = f_O(0; x) = x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots$$

This series converges in the interval $(-1, 1)$. We can, in fact, perform a Taylor expansion of function $\ln[(1 + x)/(1 - x)]$ around $x = 0$, and find that,

$$\ln\left(\frac{1+x}{1-x}\right) = 2x + \frac{2}{3}x^3 + \frac{2}{5}x^5 + \cdots$$

Therefore, Q_0 can be expressed in closed form:

$$Q_0(x) = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right) \tag{17}$$

To compute $Q_1(x)$ we will have to consider formula (16):

$$Q_1(x) = -f_E(2; x) = -\left(1 - x^2 - \frac{x^4}{3} - \frac{x^6}{5} - \cdots\right) = x\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right) - 1$$

Thus, also Q_1 can be expressed in closed form,

$$Q_1(x) = \frac{1}{2} x \ln\left(\frac{1+x}{1-x}\right) - 1 \tag{18}$$

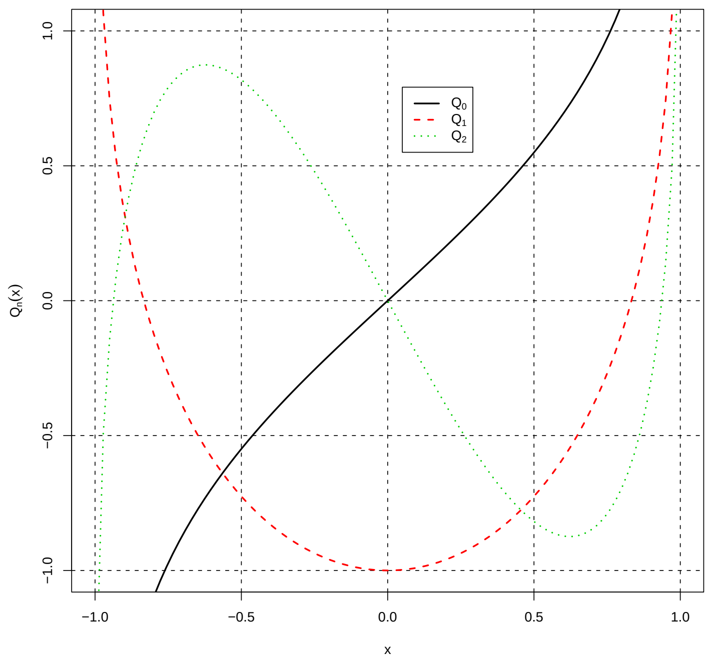

Figure 2: Plot of the first three Legendre functions of the second kind.

Indeed, all Legendre functions of the second kind have a closed form that can be derived using certain recurrence formulas. We will not described them in these notes, but they can be find easily in advanced mathematics textbooks. Plots of the first three Legendre functions of the second kind are shown in Figure 2.

EXAMPLE 5.
Express the general solution of the equation at example 3, using Legendre polynomials and Legendre functions of the second kind.
Solution
The general solution found was given by,

$$y(x) = a_0 f_E(12; x) + BP_3(x)$$

Using definition (16), with $n = 3$, we notice that $f_E(12; x) = (3/2)Q_3(x)$. Therefore we can re-write the above general solution as,

$$y(x) = AQ_3(x) + BP_3(x),$$

where A is an arbitrary constant replacing a_0.

4 Rodrigues' formula

Legendre polynomials can be computed iteratively one after the other with the aid of a formula which makes use of repeted derivatives. This formula is known as *Rodrigues' formula*, and the aim of this section is to derive it and illustrate its use.
To fix ideas let us consider definition (12) for n even:

$$P_n(x) = \sum_{m=0}^{n/2} (-1)^m \frac{(2n-2m)!}{2^n m!(n-m)!(n-2m)!} x^{n-2m}$$

If we derive x^{2n-2m} n times, we get,

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}x^{2n-2m} = (2n-2m)(2n-2m-1)\cdots(2n-2m-n+1)x^{n-2m}$$

i.e.

$$\frac{\mathrm{d}^n}{\mathrm{d}x^n}x^{2n-2m} = \frac{(2n-2m)!}{(n-2m)!}x^{n-2m}$$

Therefore, the expression for $P_n(x)$ can be re-written as,

$$P_n(x) = \frac{1}{2^n}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\sum_{m=0}^{n/2}\frac{(-1)^m}{m!(n-m)!}x^{2n-2m}\right]$$

or, multiplying and dividing by $n!$,

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\sum_{m=0}^{n/2}\frac{(-1)^m n!}{m!(n-m)!}x^{2n-2m}\right]$$

The summation used in the above expression can be extended to $m = n$, because it is easy to prove that,

$$\sum_{m=n/2+1}^{n}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\frac{(-1)^m n!}{m!(n-m)!}x^{2n-2m} = 0$$

Therefore,

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}\left[\sum_{m=0}^{n}\frac{(-1)^m n!}{m!(n-m)!}x^{2n-2m}\right]$$

Now, the expression in square brackets is just the binomial expansion of $(x^2-1)^n$:

$$(x^2-1)^n = \sum_{m=0}^{n}\begin{pmatrix} n \\ m \end{pmatrix}(x^2)^{n-m}(-1)^m = \sum_{m=0}^{n}\frac{(-1)^m n!}{m!(n-m)!}x^{2n-2m}$$

The whole content of the square brackets can, thus, be replaced by $(x^2-1)^n$. Ultimately, this leads to Rodrigues' formula:

$$P_n(x) = \frac{1}{2^n n!}\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n \tag{19}$$

Although we have derived formula (19) for n even, the same derivation can be done for n odd. Rodrigues' formula is valid for any integer value of n.

<u>EXAMPLE 6</u>.
Using Rodrigues' formula compute the first four Legendre polynomials.
<u>Solution</u>
The zeroth-order derivative of a function is simply the function itself. So,

$$P_0(x) = (x^2-1)^0 = 1$$

For $n = 1, 2, 3$ the calculation is straightforward:

$$\begin{aligned}
P_1(x) &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}(x^2-1) = x \\
P_2(x) &= \frac{1}{8}\frac{\mathrm{d}^2}{\mathrm{d}x^2}(x^2-1)^2 = \frac{1}{2}(3x^2-1) \\
P_3(x) &= \frac{1}{48}\frac{\mathrm{d}^n}{\mathrm{d}x^3}(x^2-1)^3 = \frac{1}{2}(5x^3-3x)
\end{aligned}$$

which is the same result obtained in example 2.

EXAMPLE 7.
Using Rodrigues' formula prove the following identity:

$$\int_{-1}^{1}[P_n(x)]^2\mathrm{d}x = \frac{2}{2n+1} \tag{20}$$

Solution
Using Rodrigues' formula, the above integral can be re-written as,

$$\frac{1}{2^{2n}(n!)^2}\int_{-1}^{1}\left[\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n\right]\left[\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n\right]\mathrm{d}x$$

An integration by parts yields,

$$\frac{1}{2^{2n}(n!)^2}\left\{\left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(x^2-1)^n\right]\left[\frac{\mathrm{d}^n}{\mathrm{d}x^n}(x^2-1)^n\right]\Bigg|_{-1}^{1}-\int_{-1}^{1}\left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(x^2-1)^n\right]\left[\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}(x^2-1)^n\right]\mathrm{d}x\right\}$$

$$\Downarrow$$

$$\frac{1}{2^{2n}(n!)^2}\left\{-\int_{-1}^{1}\left[\frac{\mathrm{d}^{n-1}}{\mathrm{d}x^{n-1}}(x^2-1)^n\right]\left[\frac{\mathrm{d}^{n+1}}{\mathrm{d}x^{n+1}}(x^2-1)^n\right]\mathrm{d}x\right\}$$

A second integration by parts yields,

$$\frac{1}{2^{2n}(n!)^2}\left\{(-1)^2\int_{-1}^{1}\left[\frac{\mathrm{d}^{n-2}}{\mathrm{d}x^{n-2}}(x^2-1)^n\right]\left[\frac{\mathrm{d}^{n+2}}{\mathrm{d}x^{n+2}}(x^2-1)^n\right]\mathrm{d}x\right\}$$

By carrying out integration by parts n times, eventually we will obtain,

$$\frac{(-1)^n}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n\left[\frac{\mathrm{d}^{2n}}{\mathrm{d}x^{2n}}(x^2-1)^n\right]\mathrm{d}x$$

$(x^2-1)^n$ is a polynomial, whose highest-power term is x^{2n}. Taking its derivative $2n$ times simply produces $(2n)!$. Therefore, we are left with the following integral to compute,

$$\int_{-1}^{1}[P_n(x)]^2\mathrm{d}x = \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\int_{-1}^{1}(x^2-1)^n\mathrm{d}x$$

In Appendix A it is shown that,

$$\int_{-1}^{1}(x^2-1)^n\mathrm{d}x = \frac{(-1)^n 2^{n+1}n!}{3\cdot 5\cdots(2n+1)}$$

We have, thus,

$$\begin{aligned}
\int_{-1}^{1}[P_n(x)]^2\mathrm{d}x &= \frac{(-1)^n(2n)!}{2^{2n}(n!)^2}\frac{(-1)^n2^{n+1}n!}{3\cdot 5\cdots(2n+1)} \\
&= \frac{2[(2n)(2n-1)(2n-2)(2n-3)\cdots 5\cdot 4\cdot 3\cdot 2\cdot 1]}{2^n n![(2n+1)(2n-1)(2n-3)\cdots 5\cdot 3\cdot 1]} \\
&= \frac{2[(2n)(2n-2)(2n-4)\cdots 4\cdot 2]}{2^n n!(2n+1)} \\
&= \frac{2[2^n n(n-1)(n-2)\cdots 2\cdot 1]}{2^n n!(2n+1)} \\
&= \frac{2\cdot 2^n n!}{2^n n!(2n+1)} = \frac{2}{2n+1}
\end{aligned}$$

which is what we wanted to prove.

5 Orthogonality of Legendre polynomials

As it is the case for other objects of mathematical physics, Legendre polynomials can be used for series expansion. This happens for instance to the trigonometric functions sine and cosine, found in the Fourier series for the interval $(-\infty, +\infty)$. We have also found Bessel functions $J_m(\alpha_n x)$ to constitute a valid set to decompose a function. In this case m is a fixed, arbitrary integer, while $\{\alpha_n\}$ is the infinite succession of J_m's zeroes. In a similar fashion, the infinite set of Legendre polynomials, $\{P_n(x)\}$, can be used to expand a function as a series of the type,

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x)$$

in the interval $[-1, +1]$. To show how this is possible, we need first to demonstrate that Legendre polynomials constitute an orthogonal set. In short, a set of infinite functions, $\{\varphi_0(x), \varphi_1(x), \varphi_2(x), \ldots\}$, is said to be *orthogonal* in a given interval (a, b) if the following property is verified:

$$\int_a^b \varphi_n(x)\varphi_m(x)p(x)\mathrm{d}x = \begin{cases} 0 & \text{if} \quad n \neq m \\ K_n & \text{if} \quad n = m \end{cases}$$

where K_n is a constant. We are now going to show that Legendre polynomials are just such a set of orthogonal functions, because they obey the following property:

$$\int_{-1}^{1} P_n(x)P_m(x)\mathrm{d}x = \begin{cases} 0 & \text{if} \quad n \neq m \\ 2/(2n+1) & \text{if} \quad n = m \end{cases} \tag{21}$$

(It is easily seen that, for Legendre polynomials, the orthogonality refers to the interval $[-1, 1]$, and the weight function is $p(x) = 1$).
To show property (21) is true, we have to consider that any polynomial $P_n(x)$ obeys Legendre's equation, that is,

$$(1 - x^2)P_n'' - 2xP_n' + n(n+1)P_n = 0,$$

which can also be written as,

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1 - x^2)P_n'\right] + n(n+1)P_n = 0$$

Multiplying the above expression by P_m, and integrating between -1 and +1, we obtain:

$$\int_{-1}^{1} P_m \frac{\mathrm{d}}{\mathrm{d}x}\left[(1 - x^2)P_n'\right]\mathrm{d}x + n(n+1)\int_{-1}^{1} P_m P_n \mathrm{d}x = 0$$

An integration by parts yields,

$$\left[(1 - x^2)P_n' P_m\right]_{-1}^{1} - \int_{-1}^{1}(1 - x^2)P_n' P_m' \mathrm{d}x + n(n+1)\int_{-1}^{1} P_m P_n \mathrm{d}x = 0$$

$$\Downarrow$$

$$n(n+1)\int_{-1}^{1} P_m P_n \mathrm{d}x = \int_{-1}^{1}(1 - x^2)P_n' P_m' \mathrm{d}x \tag{22}$$

With an analogous series of passages, starting from the equation for P_m (with $m \neq n$) and successively multiplying by P_n, we obtain:

$$m(m+1)\int_{-1}^{1} P_n P_m \mathrm{d}x = \int_{-1}^{1}(1 - x^2)P_m' P_n' \mathrm{d}x \tag{23}$$

At this point it will suffice to subtract equation (23) from equation (22), to obtain this relationship,

$$[n(n+1) - m(m+1)] \int_{-1}^{1} P_n(x)P_m(x)\mathrm{d}x = 0,$$

which is true only if

$$\int_{-1}^{1} P_n(x)P_m(x)\mathrm{d}x = 0$$

This verifies the first of (21). To verify the second, we simply take $m = n$ and, thus, have to compute the following integral

$$\int_{-1}^{1} [Pn(x)]^2 \mathrm{d}x$$

Indeed, this has already been done at example 7 (result (20)). The integral yields,

$$\int_{-1}^{1} [Pn(x)]^2 \mathrm{d}x = \frac{2}{2n+1}$$

Thus, the second part of equation (21) is also true.

EXAMPLE 8.
Obtain the formula to compute the expansion coefficient for a Legendre series.
Solution
Given a function $f(x)$, behaving sufficiently well in the interval $[-1, 1]$ (i.e. a piecewise, continuous function), its series expansion using Legendre polynomials can be written as

$$f(x) = \sum_{n=0}^{\infty} a_n P_n(x) \tag{24}$$

Let us multiply both sides of the above equation by $P_m(x)$, and integrate between -1 and 1:

$$\int_{-1}^{1} f(x)P_m(x)\mathrm{d}x = \sum_{n=0}^{\infty} a_n \int_{-1}^{1} P_n(x)P_m(x)\mathrm{d}x$$

Using the orthogonality between P_n and Pm, the only component of the infinite series to be different from zero is the one with $n = m$. Thus,

$$\int_{-1}^{1} f(x)P_m(x)\mathrm{d}x = a_m \frac{2}{2m+1}$$

This last expression can be re-written with n instead of m, because m is a dummy variable,

$$a_n = \frac{2n+1}{2} \int_{-1}^{1} f(x)P_n(x)\mathrm{d}x \tag{25}$$

EXAMPLE 9.
Find Legendre series for the polynomial $f(x) = 4x + 3x^2 - 5x^3$.
Solution
It is quite obvious that any polynomial of degree n can be expressed as a linear combination of Legendre polynomials up to the n-th degree. For instance, $1 = P_0(x)$, $x = P_1(x)$, $x^2 = (2/3)P_2(x) + (1/3)P_0(x)$, and so on. Given that the decomposition of a function according to an orthogonal set is unique, then any polynomial of degree n is a combination of Legendre polynomials up to the n-th one. In this case the function is a polynomial of degree 3, therefore

it will be a combination of Legendre polynomials P_0, P_1, P_2 and P_3. To compute the coefficients of the linear combination, formula (25) can be used four times:

$$\begin{aligned}
a_0 &= \frac{1}{2}\int_{-1}^{1}(4x+3x^2-5x^3)\mathrm{d}x = 1 \\
a_1 &= \frac{3}{2}\int_{-1}^{1}(4x+3x^2-5x^3)x\mathrm{d}x = 1 \\
a_2 &= \frac{5}{2}\int_{-1}^{1}(4x+3x^2-5x^3)\left[\frac{1}{2}(3x^2-1)\right]\mathrm{d}x = 2 \\
a_3 &= \frac{7}{2}\int_{-1}^{1}(4x+3x^2-5x^3)\left[[\frac{1}{2}(5x^3-3x)\right]\mathrm{d}x = -2
\end{aligned}$$

It can be verified, as it should be, that $a_4 = 0$, $a_5 = 0$, and so on. The expansion we were looking for is, therefore,

$$4x + 3x^2 - 5x^3 = P_0(x) + P_1(x) + 2P_2(x) - 2P_3(x)$$

6 A boundary value problem for Laplace's equation. Steady-state temperature inside a sphere

A sphere of radius a, made up of a homogeneous material, is centred at the origin of a reference system. The surface of its top half is maintained at a constant temperature ξ, while the surface of its bottom half is kept at the constant temperature of zero degrees centigrades. If we wait long enough after the top and bottom halves are put in contact with the heating sources, the temperature distribution inside the sphere will reach a stationary state, described by Laplace's equation. In mathematical terms this boundary value problem can be described by the following set of equations,

$$\begin{cases} \nabla^2 u = 0 \\ u(a,\theta,\phi) = \xi & \text{if} \quad 0 \le \theta < \pi/2 \\ u(a,\theta,\phi) = 0 & \text{if} \quad \pi/2 \le \theta \le \pi \end{cases} \tag{26}$$

Laplace's equation in spherical coordintes is:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0$$

It can be solved by separation of variables. First, a factorised solution is postulated,

$$u(r,\theta,\phi) = R(r)T(\theta)F(\phi) \tag{27}$$

Then this is replaced into the equation, yielding,

$$\frac{TF}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{RF}{r^2\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}T}{\mathrm{d}\theta}\right) + \frac{RT}{r^2\sin^2\theta}\frac{\mathrm{d}^2F}{\mathrm{d}\phi^2} = 0$$

Once the obtained result is divided by RTF and multiplied by $r^2\sin^2\theta$, we obtain,

$$\frac{\sin^2\theta}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{\sin\theta}{T}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}T}{\mathrm{d}\theta}\right) = -\frac{1}{F}\frac{\mathrm{d}^2F}{\mathrm{d}\phi^2} \tag{28}$$

The left-hand side of the above equation depends only on r and θ, while its right-hand side depends only on ϕ. The equation can be satisfied only if both sides are equal to a constant which, given that ϕ is an angular variable, it is better to choose as a positive one, m^2. From (28) the following equation is,thus, derived for ϕ:

$$\frac{\mathrm{d}^2F}{\mathrm{d}\phi^2} + m^2F = 0 \tag{29}$$

Its solution is readily written as,

$$F(\phi) = k_1 \sin(m\phi) + k_2 \cos(m\phi)$$

The boundary conditions do not depend on ϕ. This can be realised only if m is set equal to zero. Therefore,

$$F(\phi) = k_2 \tag{30}$$

Let us now solve the rest of the equation by selecting the left-hand side of (28) and $m = 0$:

$$\frac{\sin^2\theta}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) + \frac{\sin\theta}{T}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}T}{\mathrm{d}\theta}\right) = 0$$

$$\Downarrow$$

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) = -\frac{1}{T\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}T}{\mathrm{d}\theta}\right) \tag{31}$$

Given that the left-hand side of (31) depends on r, while its right-hand side depends on θ, the equation can be satisfied only if both members equal a constant which temporarily can be any real number, α. Equating the right-hand side of (31) to this constant, we obtain the following equation:

$$\frac{1}{\sin\theta}\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin\theta\frac{\mathrm{d}T}{\mathrm{d}\theta}\right) + \alpha T = 0 \tag{32}$$

It can be transformed into Legendre's equation if the following variable substitution is adopted:

$$x \equiv \cos\theta$$

With this substitution we have $\mathrm{d}/\mathrm{d}\theta = -\sin\theta\mathrm{d}/\mathrm{d}x = -\sqrt{1-x^2}\mathrm{d}/\mathrm{d}x$ (you might be arguing that $sin\theta = \pm\sqrt{1-\cos\theta} \equiv \pm\sqrt{1-x^2}$; but θ varies between 0 and π, where the sine is positive, therefore $sin\theta = \sqrt{1-x^2}$). Equation (32) is, thus, replaced by,

$$\frac{1}{\sqrt{1-x^2}}\left\{-\sqrt{1-x^2}\frac{\mathrm{d}}{\mathrm{d}x}\left[-(1-x^2)\frac{\mathrm{d}T}{\mathrm{d}x}\right]\right\} + \alpha T = 0$$

$$\Downarrow$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}T}{\mathrm{d}x}\right] + \alpha T = 0$$

And, finally,

$$(1-x^2)T'' - 2xT' + \alpha T = 0$$

where prime and double prime indicate first and second derivative with respect to x. As promised, this is the Legendre's equation. We require finite solutions at $x = \pm1$ (corresponding to $\theta = 0, \pi$). Therefore the constant α will have to be equal to $\ell(\ell+1)$, where ℓ is an integer. In such a case any Legendre polynomial will be a solution of the equation,

$$T(\theta) = c_\ell P_\ell(\cos\theta), \tag{33}$$

To complete the solution of our boundary value problem let us equate the right-hand side of (31) to $\alpha \equiv \ell(\ell+1)$:

$$\frac{1}{R}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}R}{\mathrm{d}r}\right) = \ell(\ell+1)$$

$$\Downarrow$$

$$r^2\frac{\mathrm{d}^2R}{\mathrm{d}r^2} + 2r\frac{\mathrm{d}R}{\mathrm{d}r} - \ell(\ell+1)R = 0 \tag{34}$$

Equation (34) is a particular case of a class of differential equations known as *Euler-Cauchy equations*. Without going into too much details, this equation can be solved by postulating a solution of the form $R(r) = r^\beta$. If this form is used, equation (34) becomes,

$$\beta(\beta-1) + 2\beta - \ell(\ell+1) = 0$$

which has two solutions, $\beta = \ell$ and $\beta = -(\ell+1)$. Consequently, two independent solutions of equation (34) are $1/r^{\ell+1}$ and r^ℓ. The first solution is not physically acceptable, because it becomes infinite at $r = 0$, inside the sphere. Thus, the solution we accept is,

$$R(r) = b_\ell r^\ell \tag{35}$$

A solution of Laplace's equation inside the sphere is built by multiplying (30), (33) and (35):

$$u_\ell(r,\theta,\phi) = b_\ell r^\ell c_\ell P_\ell(\cos\theta) k_2 \equiv h_\ell r^\ell P_\ell(\cos\theta)$$

The general solution is, therefore,

$$u(r,\theta,\phi) = \sum_{\ell=0}^{\infty} h_\ell r^\ell P_\ell(\cos\theta) \tag{36}$$

To compute coefficients h_ℓ, we simply use the boundary conditions (26), which can be here reported as,

$$u(a,\theta,\phi) = f(\theta) = \begin{cases} \xi & if \quad 0 \le \theta < \pi/2 \\ 0 & if \quad \pi/2 \le \theta \le \pi \end{cases}$$

Applying these to expression (36) yields,

$$\sum_{\ell=0}^{\infty} h_\ell a^\ell P_\ell(\cos\theta) = f(\theta)$$

We can, thus, compute each Legendre polynomial's coefficient, $h_\ell a^\ell$, using formula (25), which here becomes,

$$h_\ell a^\ell = \frac{2\ell+1}{2}\int_1^1 f(x)P_\ell(x)\mathrm{d}x = \frac{2\ell+1}{2}\int_0^1 \xi P_\ell(x)\mathrm{d}x = \xi\frac{2\ell+1}{2}\int_0^1 P_\ell(x)\mathrm{d}x$$

$$\Downarrow$$

$$h_\ell = \frac{\xi}{a^\ell}\frac{2\ell+1}{2}\int_0^1 P_\ell(x)\mathrm{d}x$$

Using tabulated expressions for the first six Legendre polynomials we have, for instance,

$$\begin{aligned} h_0 &= \xi/2 \\ h_1 &= 3\xi/(4a) \\ h_2 &= 0 \\ h_3 &= -7\xi/(16a^3) \\ h_4 &= 0 \\ h_5 &= 11\xi/(32a^5) \end{aligned}$$

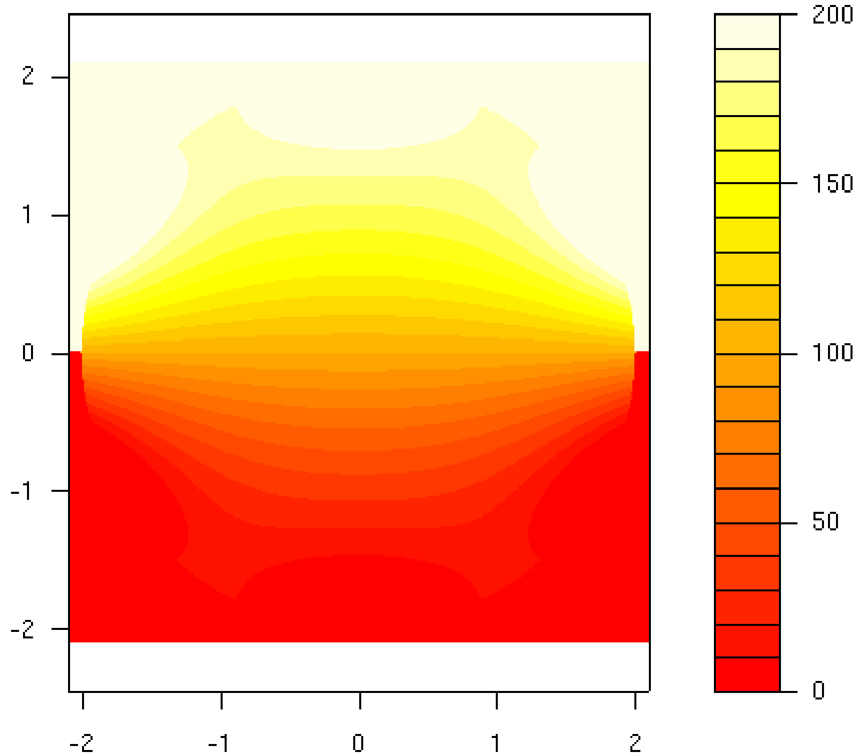

Figure 3: Temperature distribution in a section of the sphere, for the boundary value problem described in the main text.

The approximated solution to boundary value problem (26) is, finally,

$$u(r,\theta,\phi) \approx \xi \left[\frac{1}{2} P_0(\cos\theta) + \frac{3}{4}\frac{r}{a} P_1(\cos\theta) - \frac{7}{16}\left(\frac{r}{a}\right)^3 P_3(\cos\theta) + \frac{11}{32}\left(\frac{r}{a}\right)^5 P_5(\cos\theta) \right] \qquad (37)$$

This approximation is represented at Figure 3, where only a section of the sphere is shown, due to symmetry.

Appendix

A Derivation of $\int_{-1}^{1}(x^2-1)^n\mathrm{d}x$

Let us replace the integration variable x with $\cos\theta = x$. The integral becomes,

$$\int_{-1}^{1}(x^2-1)^n\mathrm{d}x = 2(-1)^n\int_{0}^{\pi/2}(\sin\theta)^{2n+1}\mathrm{d}\theta$$

Now, if we define $S(k)$ as $\int_0^{\pi/2}(\sin\theta)^k\mathrm{d}\theta$, an integration by parts leads to the following recurrence property:

$$S(k) = \frac{k-1}{k}S(k-2) \tag{38}$$

Let us now apply property (38) repeatedly for $k = 2n+1$:

$$\begin{aligned}
S(2n+1) &= \frac{2n}{2n+1}S(2n-1) \\
&= \frac{(2n)(2n-2)}{(2n+1)(2n-1)}S(2n-3) \\
&= \cdots \\
&= \frac{(2n)(2n-2)\cdots 2}{(2n+1)(2n-1)\cdots 3}S(1) \\
&= \frac{2^n n!}{3\cdot 5\cdots(2n+1)}
\end{aligned}$$

In conclusion, we have,

$$\int_{-1}^{1}(x^2-1)^n\mathrm{d}x = 2(-1)^n S(2n+1) = 2(-1)^n\frac{2^n n!}{3\cdot 5\cdots(2n+1)} = \frac{(-1)^n 2^{n+1} n!}{3\cdot 5\cdots(2n+1)}$$

which is the result used in example 7.