Solved problems for Mathematics IV tutorials

1. A given temperature field, $T(x, y, z)$, is described by the following expression:

$$T(x, y, z) = x^2 - y^2 + xyz + 273$$

   i. In which direction is the maximum rate of increase of temperature at the point $(-1, 2, 3)$?

   ii. If x, y, z are measured in metres and T in °C, what is the the maximum rate of temperature increase?

   iii. Show that the rate of temperature increase in the direction of the vector $(4, 0, 1)$ is $14/3$.

SOLUTION

(i) A scalar field has its maximum variation along a direction parallel to the gradient of the field. The gradient of $T(x, y, z)$, at a generic point (x, y, z), is the following vector field:

$$\nabla T(x, y, z) \equiv \left(\frac{\partial T}{\partial x} \frac{\partial T}{\partial y} \frac{\partial T}{\partial z} \right) = (2x + yz, -2y + xz, xy)$$

To find the direction we are looking for at point $(-1, 2, 3)$ we simply replace $x = -1$, $y = 2$, $z = 3$ in this expression. The result is vector $(4, -7, -2)$, which gives the direction for the maximum rate of increase. Bear in mind that, although this vector describes a direction in a unique way, it is not a unit vector.

(ii) Consider the direction along which there is maximum rate of increase. This is given by a unit vector, parallel to $(4, -7, -2)$, that is by $(4, -7, -2)/|(4, -7, -2)| = (4, -7, -2)/\sqrt{69}$. A direction is a straight line; in our case it can be described by the following parametric equations,

$$\left\{ \begin{array}{ccl} x & = & -1 + (4/\sqrt{69})s \\ y & = & 2 - (7/\sqrt{69})s \\ z & = & 3 - (2/\sqrt{69})s \end{array} \right. ,$$

where the parameter s coincides with the arc length. The rate of increase itself is given by,

$$\frac{\mathrm{d}T}{\mathrm{d}s} = \frac{\partial T}{\partial x}\frac{\mathrm{d}x}{\mathrm{d}s} + \frac{\partial T}{\partial y}\frac{\mathrm{d}y}{\mathrm{d}s} + \frac{\partial T}{\partial z}\frac{\mathrm{d}z}{\mathrm{d}s}$$

$$\Downarrow$$

$$\frac{\mathrm{d}T}{\mathrm{d}s} = [\nabla T(-1,2,3)] \cdot \left[\frac{(4,-7,-2)}{\sqrt{69}}\right] = \frac{(4,-7,-2)\cdot(4,-7,-2)}{\sqrt{69}} \equiv |\nabla T(-1,2,3)| = \sqrt{69}$$

So, the maximum rate of increase is $\sqrt{69} \approx 8.307$ °C/m.

(iii) The rate of increase along a direction given by the unit vector $\mathbf{n} \equiv (n_x, n_y, n_z)$ is found projecting the gradient along it:

$$\frac{\mathrm{d}T}{\mathrm{d}n} = \mathbf{n} \cdot \nabla T$$

In the present problem we are given the direction through vector $(4,0,1)$, which has not unit length. Its norm is 3, therefore $(4,0,1)/3$ is the unit vector we need to proceed. The rate of increase is, thus, given by $[(4,0,1)/3] \cdot (4,-7,-2) = 14/3 \approx 4.667$. This value is smaller than the 8.307 previously found, as it should be, because the gradient direction is the direction with highest variation.

2. If $\phi(x,y,z) = xy + z^2$ is the scalar potential of a vector field, find the work done by the field along the path described by the following parametric equations:

$$\begin{cases} x &= t \\ y &= t^2 \\ z &= 1 \end{cases}$$

from $(0,0,1)$ to $(1,1,1)$,

i. by evaluating a line integral
ii. without evaluating a line integral

SOLUTION

The vector field generated by a scalar potential ϕ is simply its gradient, $\mathbf{F} = \nabla\phi$. Thus, through calculation of a gradient, we arrive at,

$$\mathbf{F} = (y, x, 2z)$$

(i) The expression for the work done by the field as a line integral is as follows:

$$W = \int_{A\ \gamma}^{B} \mathbf{F} \cdot \mathrm{d}\mathbf{l}$$

where A and B are the initial and final points on the integration line γ. The key quantity in the above expression is $\mathbf{F} \cdot \mathrm{d}\mathbf{l}$, essentially a dot product between a vector field and a line element, itself a vector. This vector is always tangent to the integration line. Thus, the expression $\mathrm{d}\mathbf{l}$ really means,

$$\mathrm{d}(x(t), y(t), z(t)) = (x'(t), y'(t), z'(t))\mathrm{d}t = (1, 2t, 0)\mathrm{d}t$$

Next, we already know the expression for $\mathbf{F}$, but this has to be taken along the integration line and, therefore, it needs to be parametrized using t:

$$\mathbf{F} \equiv (t^2, t, 2)$$

We have, now, all the elements to write down the line integral. Because all vectors have been parametrised using only one parameter, t, the line integral becomes a standard integral in one variable. The initial integration point, $(0,0,1)$ corresponds to $t = 0$ (remember, in the parametrisation used in this problem, $x = t$), while the final integration point corresponds to $t = 1$. So:

$$W = \int_0^1 (t^2, t, 2) \cdot (1, 2t, 0)\mathrm{d}t = \int_0^1 3t^2 \mathrm{d}t = 1$$

(ii) The vector field $\mathbf{F}$ is the gradient of a potential. This meand that it is a conservative force field and, therefore, the work done between two points A and B is the same, irrespective of the path used to go from A to B. To be more precise, this work is simply given by the difference of the potential field calculated at B and A:

$$W = \phi(B) - \phi(A)$$

In our case $A \equiv (0,0,1)$ and $B \equiv (1,1,1)$. Thus,

$$W = \phi(1,1,1) - \phi(0,0,1) = (1+1) - (0+1) = 1$$

This value is the same previously computed through a line integral, as it should be.

3. Evaluate the line integral for the vector field $\mathbf{F} \equiv (2xz, 2, x^2)$, from $A \equiv (0,0,0)$ to $B \equiv (1,0,0)$, along the following paths:

   i. the arc of the circle centred on $(1/2, 0, 0)$, laying on the $z = 0$ plane and having radius $1/2$;

   ii. the arc of the circle centred on $(1/2, 0, 0)$, laying on the $y = 0$ plane and having radius $1/2$;

   iii. the x-axis.

SOLUTION

In this problem the starting and final points, A and B, are kept unchanged; only the paths connecting them change. Therefore we have to compute three line integrals along three different paths. The first thing to do is to find out the parametric equation of each path, and then proceed with the standard evaluation of a line integral.

(i) A well known parametric representation of a plane circle centred on (α, β) and with radius R is the following:

$$\begin{cases} x &= \alpha + R\cos t \\ y &= \beta + R\sin t \end{cases}$$

The line integral we are considering here goes from A to B along an arc of a circle laying on the $z = 0$ plane. The parametric equations will, then, be:

$$\mathbf{l} = \begin{cases} x &= 1/2 + (1/2)\cos t \\ y &= (1/2)\sin t \\ z &= 0 \end{cases}$$

The line element $d\mathbf{l}$ can be immediately computed starting from this curve parametric equation:

$$\mathrm{d}\mathbf{l} = \mathrm{d}(1/2 + (1/2)\cos t, (1/2)\sin t, 0) = (-\sin t/2, \cos t/2, 0)$$

The vector field components have to be limited exclusively on points of curve $\mathbf{l}$; this is achieved simply by replacing x, y, z with their parametric expressions on the curve:

$$\begin{aligned}
\mathbf{F} &\equiv (2xz, 2, x^2) \\
&= (2 \cdot (1/2)(1+\cos t) \cdot 0, 2, (1/4)(1 + 2\cos t + cos^2 t)) \\
&= (0, 2, 1/4 + \cos t/2 + \cos^2 t/4)
\end{aligned}$$

Now we can multiply scalarly $\mathbf{F}$ and $\mathrm{d}\mathbf{l}$:

$$\mathbf{F} \cdot \mathrm{d}\mathbf{l} = \cos t$$

Point A on the curve corresponds to $t = \pi$, while point B to $t = 0$ (to see this, for example for A, consider that $(1/2 + (1/2)\cos\pi, (1/2)\sin\pi, 0) = (0, 0, 0)$). The line integral is, finally, transformed into the following standard integral:

$$\int_{A\ \mathbf{l}}^{B} \mathbf{F} \cdot \mathrm{d}\mathbf{l} = \int_{\pi}^{0} \cos t \ \mathrm{d}t = 0$$

(ii) The parametric equation for the curve in this case are:

$$\mathbf{l} = \begin{cases} x = 1/2 + (1/2)\cos t \\ y = 0 \\ z = (1/2)\sin t \end{cases}$$

From them we can calculate the following line element:

$$\mathrm{d}\mathbf{l} = \mathrm{d}(1/2 + (1/2)\cos t, 0, (1/2)\sin t) = (-\sin t/2, 0, \cos t/2)$$

The vector field on this curve is parametrized as follows:

$$\mathbf{F} \equiv (\sin t(1+\cos t)/2, 2, (1 + 2\cos t + \cos^2 t)/4)$$

In the end the line integral yields:

$$\begin{aligned}
\int_{A\ \mathbf{l}}^{B} \mathbf{F} \cdot \mathrm{d}\mathbf{l} &= \int_{\pi}^{0} \left[-\frac{1}{4}\sin^2 t(1+\cos t) + \frac{1}{8}\cos t(1 + 2\cos t + \cos^2 t) \right] \mathrm{d}t \\
&= \int_{\pi}^{0} \left[-\frac{1}{4} + \frac{1}{2}\cos^2 t - \frac{3}{8}\sin^2 t\cos t + \frac{1}{4}\cos t \right] \mathrm{d}t \\
&= \left[-\frac{1}{4}t + \frac{1}{4}\left(t + \frac{\sin 2t}{2}\right) - \frac{\sin^3 t}{8} + \frac{1}{4}\sin t \right]_{\pi}^{0} = 0
\end{aligned}$$

(iii) The straight line for this case is the x-axis, whose parametric equations are, for example,

$$\mathbf{l} = \begin{cases} x = t \\ y = 0 \\ z = 0 \end{cases}$$

The line element is, thus, simply $d\mathbf{l} \equiv (1,0,0)dt$. Given that y and z are both zero, the vector field along this curve has components $\mathbf{F} \equiv (0, 2, t^2)$. The line integral for this path is:

$$\int_{A\ 1}^{B} \mathbf{F} \cdot d\mathbf{l} = \int_0^1 0 \; dt = 0$$

It is of no surprise that we found the same answer for all 3 cases. It can be easily verified that this vector field is the gradient of the following scalar field,

$$\phi(x, y, z) = x^2 z + 2y$$

Therefore this vector field is a conservative field, and the line integral is independent on the path used to join A to B.

4. Given the following vector field,

$$\mathbf{F} = x \sin y \mathbf{i} + \lambda \cos y \mathbf{j} + xy \mathbf{k}$$

   i. calculate div$\mathbf{F}$ at the point $(1, \pi/2, 1)$, if $\lambda = -1$;
   ii. find the value of the constant λ such that the vector field $\mathbf{F} = x \sin y \mathbf{i} + \lambda \cos y \mathbf{j} + xy \mathbf{k}$ is solenoidal.

SOLUTION

In cartesian coordinates, the divergence of a vector field $\mathbf{F} \equiv (F_x, F_y, F_z)$ is computed through the following formula:

$$\text{div}\mathbf{F} \equiv \nabla \cdot \mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

For the field assigned in the problem we have:

$$\begin{aligned} \frac{\partial F_x}{\partial x} &= \sin y \\ \frac{\partial F_y}{\partial y} &= -\lambda \sin y \\ \frac{\partial F_z}{\partial z} &= 0 \end{aligned}$$

Therefore,

$$\nabla \cdot \mathbf{F} = \sin y - \lambda \sin y = (1 - \lambda) \sin y$$

(i.) When $\lambda = -1$ the expression for the divergence is $\nabla \cdot \mathbf{F} = 2 \sin y$; by replacing in it $y = \pi/2$ we obtain $\nabla \cdot \mathbf{F}(1, \pi/2, 1) = 2$.

(ii.) A field is solenoidal when its divergence is zero. If we wish our field to be solenoidal we must impose the following condition:

$$(1 - \lambda) \sin y = 0$$

This is always true if $\lambda = 1$. Thus, $\mathbf{F} \equiv (x \sin y, \cos y, xy)$ is a solenoidal field.

5. Given the following vector field,

$$\mathbf{F} = [1 + z^2 + \lambda(x^2 - z)]\mathbf{i} + [1 + \mu(\mu - 1)xz^2]\mathbf{j} + 2xz\mathbf{k}$$

   i. calculate curl$\mathbf{F}$ at the point $(1, 1, 0)$, if $\lambda = \mu = -1$;
   ii. find the constants λ and μ such that the field is conservative.

SOLUTION

The curl of a vector field $\mathbf{F} \equiv (F_x, F_y, F_z)$ is computed, in cartesian coordinates, through the following formula:

$$\text{curl}\mathbf{F} \equiv \nabla \times \mathbf{F} \equiv \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

For the field assigned in the text we have,

$$\begin{array}{rclcrclcrcl} \frac{\partial F_x}{\partial x} & = & 2\lambda x & & \frac{\partial F_y}{\partial x} & = & \mu(\mu-1)z^2 & & \frac{\partial F_z}{\partial x} & = & 2z \\ \frac{\partial F_x}{\partial y} & = & 0 & & \frac{\partial F_y}{\partial y} & = & 0 & & \frac{\partial F_z}{\partial y} & = & 0 \\ \frac{\partial F_x}{\partial z} & = & 2z - \lambda & & \frac{\partial F_y}{\partial z} & = & 2\mu(\mu-1)xz & & \frac{\partial F_z}{\partial z} & = & 2x \end{array}$$

The curls is, thus,

$$\nabla \times \mathbf{F} \equiv \left(-2\mu(\mu - 1)xz, -\lambda, \mu(\mu - 1)z^2 \right)$$

(i.) If $\lambda = \mu = -1$, the curl becomes $(-4xz, 1, 2z^2)$. The value of this expression at point $(1, 1, 0)$ is $(0, 1, 0)$.

(ii.) When a field is conservative its curl is a null vector. The given field can, therefore, be made conservative by assigning to zero each component of its curls. In formulas:

$$\begin{cases} -2\mu(\mu - 1)xz & = & 0 \\ -\lambda & = & 0 \\ \mu(\mu - 1)z^2 & = & 0 \end{cases}$$

This system is clearly satisfied at all times if $\lambda = 0$ and $\mu = 0$ or $\mu = 1$. In other words, the vector field $\mathbf{F} \equiv (1 + z^2, 1, 2xz)$ is a conservative field.

6. Find the total outward flux of the vector field $\mathbf{F}(\mathbf{r}) = 2r\mathbf{r}$, where $\mathbf{r} = (x, y, z)$, through the surface of the sphere $x^2 + y^2 + z^2 = R^2$,

   i. by evaluating a surface integral;
   ii. by evaluating a volume integral.

SOLUTION

It is not by chance that the problem asks to carry out two different calculations to compute a same quantity. Indeed, a surface integral (a flux integral in this case) and a volume integral are connected through Gauss theorem:

$$\oint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_V \mathrm{div}\mathbf{F}\ \mathrm{d}V$$

We are free to handle all calculations using either cartesian or spherical coordinates.
In cartesian coordinates the field is given by:

$$\mathbf{F} = 2\sqrt{x^2 + y^2 + z^2}(x\mathbf{e}_x + y\mathbf{e}_y + z\mathbf{e}_z)$$

while, in spherical coordinates, it is given by:

$$\mathbf{F} = 2r^2\mathbf{e}_r$$

(in spherical coordinates the three unit vectors forming an orthonormal basis are indicated as $\mathbf{e}_r$, $\mathbf{e}_\theta$ and $\mathbf{e}_\phi$. These three versors change their cartesian components according to the point in space to which they are applied, differently to what happen for $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$. The first unit vector, $\mathbf{e}_r$ is always parallel to the radial direction and points outward; thus, $\mathbf{r} = r\mathbf{e}_r$).
The surface of the sphere is parametrized using two variables, as any other surface. In cartesian coordinates the typical parametrization is:

$$\mathbf{S} = R(\sin u \cos v\mathbf{e}_x + \sin u \sin v\mathbf{e}_y + \cos u\mathbf{e}_z)$$

with $0 \leq u \leq \pi$ and $0 \leq v < 2\pi$. In spherical coordinates the parametric representation is much simpler:

$$\mathbf{S} = R\mathbf{e}_r$$

This simplicity is, though, only apparent because, as said before, $\mathbf{e}_r$ changes cartesian components for each value of the parameters u and v. The surface element is built in the following way. First both $\partial\mathbf{S}/\partial u$ and $\partial\mathbf{S}/\partial v$ need to be computed; then their cross product will have to be worked out. Finally, the surface element, $\mathrm{d}\mathbf{S}$, is given by:

$$\mathrm{d}\mathbf{S} = \frac{\partial\mathbf{S}}{\partial u} \times \frac{\partial\mathbf{S}}{\partial v}\mathrm{d}u\mathrm{d}v$$

Let us work out the details for cartesian coordinates, first. When $\mathbf{S}$'s partial derivatives are computed we consider $\mathbf{e}_x$, $\mathbf{e}_y$ and $\mathbf{e}_z$ constants, as they do not change on the spherical surface (the cartesian versors are the only basis to have constant cartesian coordinates throughout the space). Things are going to be different for the spherical coordinates. We have, thus:

$$\frac{\partial\mathbf{S}}{\partial u} = R(\cos u \cos v\mathbf{e}_x + \cos u \sin v\mathbf{e}_y - \sin u\mathbf{e}_z)$$

$$\frac{\partial\mathbf{S}}{\partial v} = R(-\sin u \sin v\mathbf{e}_x + \sin u \cos v\mathbf{e}_y)$$

and, accordingly,

$$\frac{\partial\mathbf{S}}{\partial u} \times \frac{\partial\mathbf{S}}{\partial v} = R^2 \sin u(\sin u \cos v\mathbf{e}_x + \sin u \sin v\mathbf{e}_y + \cos u\mathbf{e}_z)$$

So, in cartesian coordinates the surface element for the given spherical surface is given by:

$$\mathrm{d}\mathbf{S} = R^2 \sin u(\sin u \cos v\mathbf{e}_x + \sin u \sin v\mathbf{e}_y + \cos u\mathbf{e}_z)\mathrm{d}u\mathrm{d}v$$

For spherical coordinates things are more difficult. The unit vectors themselves dipend on the surface parameters in the following way:

$$\begin{cases} \mathbf{e}_r & \equiv & (\sin u \cos v, \sin u \sin v, \cos u) \\ \mathbf{e}_\theta & \equiv & (\cos u \cos v, \cos u \sin v, -\sin u) \\ \mathbf{e}_\phi & \equiv & (-\sin v, \cos v, 0) \end{cases}$$

For this reason the following relations are verified:

$$\frac{\partial \mathbf{e}_r}{\partial u} = (\cos u \cos v, \cos u \sin v, -\sin u) \equiv \mathbf{e}_\theta$$

$$\frac{\partial \mathbf{e}_r}{\partial v} = (-\sin u \sin v, \sin u \cos v, 0) \equiv \sin u \mathbf{e}_\phi$$

and,

$$\frac{\partial \mathbf{S}}{\partial u} = R\frac{\partial \mathbf{e}_r}{\partial u} = R\mathbf{e}_\theta$$

$$\frac{\partial \mathbf{S}}{\partial v} = R\frac{\partial \mathbf{e}_r}{\partial v} = R\sin u \mathbf{e}_\phi$$

Finally, as $\mathbf{e}_\theta \times \mathbf{e}_\phi = \mathbf{e}_r$, the expression for the surface element in spherical coordinates is:

$$\mathrm{d}\mathbf{S} = R^2 \sin u \mathbf{e}_r \mathrm{d}u\mathrm{d}v$$

Another object we will need is the expression for the divergence. In cartesian coordinates it is very simple:

$$\mathrm{div}\mathbf{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}$$

In spherical coordinates the divergence has a somewhat more complicated form:

$$\mathrm{div}\mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta F_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}$$

The last item we have to look into is the volume element for the volume integral. This should be clear from previous courses in standard Calculus. For cartesian coordinates it is, simply,

$$\mathrm{d}V = \mathrm{d}x\mathrm{d}y\mathrm{d}z$$

For spherical coordinates it is,

$$\mathrm{d}V = r^2\sin\theta\mathrm{d}r\mathrm{d}\theta\mathrm{d}\phi$$

We have now all the ingredients to solve our problem.

(i.) The field needs to be computed on the sphere surface. Thus,

$$\mathbf{F} = 2R^2(\sin u \cos v \mathbf{e}_x + \sin u \sin v \mathbf{e}_y + \cos u \mathbf{e}_z)$$

The flux element, $\mathbf{F}\cdot\mathrm{d}\mathbf{S}$ yields:

$$\begin{aligned} &\mathbf{F}\cdot\mathrm{d}\mathbf{S} \\ &= 2R^4\sin u(\sin u\cos v\mathbf{e}_x + \sin u\sin v\mathbf{e}_y + \cos u\mathbf{e}_z)\cdot \\ &\cdot(\sin u\cos v\mathbf{e}_x + \sin u\sin v\mathbf{e}_y + \cos u\mathbf{e}_z)\mathrm{d}u\mathrm{d}v \\ &= 2R^4\sin u\mathrm{d}u\mathrm{d}v \end{aligned}$$

Remembering, now, that $0 \leq u \leq \pi$ and $0 \leq v < 2\pi$, the flux integral becomes the following double integral:

$$\oint_S \mathbf{F} \cdot \mathrm{d}\mathbf{S} = \int_0^{\pi} \int_0^{2\pi} 2R^4 \sin u \mathrm{d}u \mathrm{d}v = 8\pi R^4$$

If we use spherical coordinates, the field computed on the spherical surface is:

$$\mathbf{F} = 2R^2 \mathbf{e}_r$$

so that the flux element, $\mathbf{F} \cdot \mathrm{d}\mathbf{S}$ yields:

$$\mathbf{F} \cdot \mathrm{d}\mathbf{S} = 2R^2 \mathbf{e}_r \cdot R^2 \sin u \mathrm{d}u \mathrm{d}v \mathbf{e}_r = 2R^4 \sin u \mathrm{d}u \mathrm{d}v$$

and the double integral will give the same result as in the cartesian case.

(ii.) Although computing the divergence of $\mathbf{F}$ in cartesian coordinates, working out its volume integral inside a spherical region is far too complicated (if not impossible!). We will, therefore, skip to the details for spherical coordinates. As we know the field has only the radial component, $F_r = 2r^2$; the divergence will, thus, be simply given by:

$$\mathrm{div}\mathbf{F} = \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 F_r \right) = \frac{1}{r^2} \frac{\partial}{\partial r} (2r^4) = 8r$$

The volume integral will be a triple integral with r between 0 and R, θ between 0 and π, and ϕ between 0 and 2π:

$$\begin{aligned} \int_V \mathrm{div}\mathbf{F} \ \mathrm{d}V &= \int_0^R \int_0^{\pi} \int_0^{2\pi} 8r r^2 \sin\theta \mathrm{d}r \mathrm{d}\theta \mathrm{d}\phi \\ &= 16\pi \left(\int_0^R r^3 \mathrm{d}r \right) \left(\int_0^{\pi} \sin\theta \mathrm{d}\theta \right) \\ &= 16\pi \frac{R^4}{4} 2 = 8\pi R^4 \end{aligned}$$

This results coincides with the one previously obtained using a flux integral, thus confirming the validity of Gauss' theorem.